Sample questions for MATH4060 Final Exam

Note: I have included only questions about the part of the course that was not covered by the midterm exam. You should also consult the homework assignments, for some excellent questions about the topics we covered.

1. (a) Let

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}, \qquad t > 0.$$

   State and prove a formula relating $\vartheta(t)$ and $\vartheta(1/t)$. You may use without proof the formula for the Fourier transform of $e^{-\pi x^2}$, as well as the Poisson summation formula, but you should state clearly these results when you use them.

   (b) Explain how the result in part (a) is relevant to the analytic continuation of the Riemann ζ function. You may use, without proof, the formula

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty u^{s/2}\frac{\vartheta(u)-1}{2}\frac{du}{u},$$

   that holds when $\operatorname{Re} s > 1$.

2. (a) What is the order of growth of $1/\Gamma$ as an entire function?

   (b) State (without proof) all the zeroes of $1/\Gamma$.

   (c) Deduce the Hadamard factorization of $1/\Gamma$.

3. Find all zeroes of ζ outside the closed strip $\{z \in \mathbb{C} \colon 0 \le \operatorname{Re} z \le 1\}$. You may use without proof the product factorization of ζ, as well as the formula for analytic continuation of ζ, but you should state clearly these results when you use them.

4. (a) Show that $\Gamma(s)$ is real whenever $s \in \mathbb{R}$ and $s \notin \{0, -1, -2, -3, \dots\}$.

   (b) Show that

$$\left|\Gamma\left(\frac{1}{2} + it\right)\right| = \sqrt{\frac{2\pi}{e^{\pi t} + e^{-\pi t}}} \quad \text{for all } t \in \mathbb{R}.$$

   (c) Hence, or otherwise, compute the value of $\Gamma(1/2)$.

5. (a) State and prove Montel's theorem about normal families of holomorphic functions.

   (b) Explain how Montel's theorem is relevant in the proof of the Riemann mapping theorem.

6. Let $\mathcal{F}$ be the family of holomorphic functions given by

$$\mathcal{F} = \{f \colon \mathbb{D} \to \mathbb{D} \text{ holomorphic } : f'(0) = 1/2\}.$$

   Show that there exists $g \in \mathcal{F}$ such that

$$g(0) = \sup_{f \in \mathcal{F}} |f(0)|.$$